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LETTER TO THE EDITOR

A note on twisting type-N solutions

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Abstract. It is shown that the twisting type-N equations for empty space derived by Hauser are a special case of the equations given by Robinson *et al.*

A family of algebraically special vacuum metrics in which the degenerate principal null direction is twisting was presented by Robinson and co-workers (Robinson and Robinson 1969, Robinson *et al* 1969). In his survey of exact solutions of the Einstein field equations, Kinnersley (1975) stated that an assumption which leads to the Robinson *et al* metrics is that the Riemann tensor falls off asymptotically as ρ^{-3}, thus excluding radiative solutions. Robinson (1975) has shown that solutions may be obtained from Robinson *et al* in which the Riemann tensor falls off like ρ^{-2}. In fact, among the Robinson *et al* metrics are some for which the Riemann tensor falls off like ρ^{-1}, although these are not explicitly exhibited. These metrics are therefore of type N in the Petrov classification and contain Hauser's (1974, 1978) one-parameter family of twisting type-N gravitational fields.

Robinson *et al* have shown that, if a space–time admits a null vector field k^i ($i = 1, 2, 3, 4$) tangent to a shear-free diverging congruence of affinely parametrised null geodesic curves, then coordinates $x^i = (\zeta, \bar{\zeta}, \sigma, \rho)$ may be chosen such that the line-element built around the congruence has the form

$$ds^2 = 2P\bar{P}\,d\zeta\,d\bar{\zeta} + 2\,d\Sigma(d\rho + Z\,d\zeta + \bar{Z}\,d\bar{\zeta} + S\,d\Sigma), \tag{1a}$$

$$d\Sigma = a(\Omega\,d\zeta + \bar{\Omega}\,d\bar{\zeta} + d\sigma) = k_i\,dx^i, \tag{1b}$$

with a and Ω independent of ρ, and a bar denoting complex conjugation. For any function $f(\zeta, \bar{\zeta}, \sigma)$ we use the notation

$$Df = \partial f/\partial\zeta - \Omega\dot{f}, \qquad \bar{D}f = \partial f/\partial\bar{\zeta} - \bar{\Omega}\dot{f}, \tag{2}$$

where a dot indicates partial differentiation with respect to σ.

The vacuum field equations for type-N solutions are satisfied provided

$$P = e^u(\rho + i\omega), \tag{3a}$$

$$Z = \rho\Lambda - i(D + \Lambda)\omega, \tag{3b}$$

$$S = a^{-1}\rho\dot{u} - \tfrac{1}{2}K, \tag{3c}$$

with $u = u(\zeta, \bar{\zeta}, \sigma)$ and

$$\omega = \tfrac{1}{2}ia\,e^{-2u}(D\bar{\Omega} - \bar{D}\Omega), \tag{4a}$$

$$\Lambda = \mathrm{D}(\log a) - \dot{\Omega}, \tag{4b}$$

$$K = e^{-2u}[\bar{\mathrm{D}}(\Lambda - \mathrm{D}u) + \mathrm{D}(\bar{\Lambda} - \bar{\mathrm{D}}u)]. \tag{4c}$$

If we also define

$$J = (\mathrm{D} + \Lambda - \mathrm{D}u)(\Lambda - \mathrm{D}u), \tag{5}$$

then the basic functions a, Ω, $\bar{\Omega}$, u must satisfy

$$K\omega + \tfrac{1}{2} e^{-2u}[(\bar{\mathrm{D}} + \bar{\Lambda})(\mathrm{D} + \Lambda) + (\mathrm{D} + \Lambda)(\bar{\mathrm{D}} + \bar{\Lambda})]\omega = 0, \tag{6a}$$

$$\bar{\mathrm{D}}J = 0 = \mathrm{D}\bar{J}. \tag{6b}$$

Introducing the function $U(\zeta, \bar{\zeta}, \sigma)$ satisfying

$$\dot{U} = a\, e^{-u}, \tag{7}$$

it can be shown that (5) and (6a) are equivalent to

$$aJ = e^{u}(\mathrm{D}^2 U)^{\cdot}, \tag{8a}$$

$$(\bar{\mathrm{D}}^2\mathrm{D}^2 - \mathrm{D}^2\bar{\mathrm{D}}^2)U = 0 \tag{8b}$$

and, provided (6b) is satisfied, one can prove that

$$(\bar{\mathrm{D}}^2\mathrm{D}^2 U)^{\cdot} = (\mathrm{D}^2\bar{\mathrm{D}}^2 U)^{\cdot} = a\, e^{-u} J\bar{J}, \tag{9}$$

thus showing that the left-hand side of (8b) is independent of σ.

The twist of the vector field (1b) is given by $-\omega/(\rho^2 + \omega^2)$. Clearly the twist vanishes if $\omega = 0$. The only non-vanishing tetrad components of the Riemann tensor is, in Newman and Penrose (1966) notation,

$$\Psi_4 = [-e^{-2u}/(\rho + i\omega)]a^{-1}\dot{J}. \tag{10}$$

The coordinate transformations which leave invariant the form (1a) of the metric are products of the trivial transformation

$$\zeta \to \bar{\zeta}, \qquad \bar{\zeta} \to \zeta, \qquad \rho \to \rho, \qquad \sigma \to \sigma \tag{11}$$

with

$$\zeta \to f(\zeta), \qquad \bar{\zeta} \to \bar{f}(\bar{\zeta}), \qquad \rho \to \rho, \qquad \sigma \to \sigma \tag{12}$$

and

$$\zeta \to \zeta, \qquad \bar{\zeta} \to \bar{\zeta}, \qquad \rho \to \rho g(\zeta, \bar{\zeta}, \sigma), \qquad \sigma \to h(\zeta, \bar{\zeta}, \sigma), \tag{13}$$

where f', $\bar{f}'$, g and $\dot{h}$ are non-zero. The first transformation is essentially complex conjugation. Invariance under the other two follows directly from the invariance of

$$P\bar{P}\, d\zeta\, d\bar{\zeta}, \qquad \rho\, d\Sigma, \qquad \rho^{-1}(d\rho + Z\, d\zeta + \bar{Z}\, d\bar{\zeta} + S\, d\Sigma) \tag{14}$$

under (12) and (13).

We can use (12) and (13) to eliminate some of the functions a, Ω, $\bar{\Omega}$, u, restricting the transformations at our disposal accordingly. If ω vanishes we can take $\Omega = \bar{\Omega} = 0$. Under the transformations (12) and (13)

$$a \to a/g\dot{h}, \qquad u \to u - \tfrac{1}{2}\log(f'\bar{f}'g^2). \tag{15}$$

Clearly we can specialise the coordinates ρ and σ so that

$$a = 1 \qquad \text{with } g\dot{h} = 1 \tag{16}$$

or

$$u = 0 \qquad \text{with } g^2 f' \bar{f}' = 1 \tag{17}$$

or both.

The type-N field equations ($6a$) and ($6b$) correspond to equations (20) and (21) respectively of Hauser (1978). Imposing condition (17) we find that the two pairs of equations are identical, with Hauser's (1978) D, D*, Δ, $\mathscr{D}$, $\mathscr{D}^*$, p and h corresponding to our D, $\bar{\mathrm{D}}$, $a^{-1}\omega$, $\mathrm{D} - \dot{\Omega}$, $\bar{\mathrm{D}} - \mathring{\Omega}$, a and $-J$ respectively. Hauser's (1974, 1978) twisting type-N solution (the only one presently known) was obtained by choosing $\Omega = \mathrm{i}(\zeta + \bar{\zeta})$ and computing the successive integrability conditions on the type-N field equations. This yields for the function a which appears in ($1b$) the solution $a = (\xi^2)^{3/4} f(y)$, where $\sqrt{2}\,\xi = \zeta + \bar{\zeta}$, $y = \sigma/\xi^2$ and $f(y)$ satisfies the equation

$$\mathrm{d}^2 f/\mathrm{d}y^2 + 3f/16(1 + y^2) = 0. \tag{18}$$

Finally, imposing the further restriction (16) on the metric, the type-N field equations ($6a$) and ($6b$) reduce to the relatively simple form

$$\bar{\mathrm{D}} \frac{\partial}{\partial \sigma}(\mathrm{D}\Omega) = 0, \tag{19a}$$

$$(\bar{\mathrm{D}}^2 \mathrm{D} - \mathrm{D}^2 \bar{\mathrm{D}})\Omega = 0. \tag{19b}$$

Equation (49) of Kinnersley (1975) contains type-N field equations which are due to A Exton (private communication). The first of these agrees with ($19a$). The second, however, is different from ($19b$), being of lower order.

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References

Hauser I 1974 *Phys. Rev. Lett.* **33** 1112
—— 1978 *J. Math. Phys.* **19** 661
Kinnersley W 1975 in *General Relativity and Gravitation* ed. G Shaviv and J Rosen (New York: Wiley) pp 109–35
Newman E T and Penrose R 1966 *J. Math. Phys.* **7** 863
Robinson I 1975 *Gen. Rel. Grav.* **6** 423
Robinson I and Robinson J R 1969 *Int. J. Theor. Phys.* **2** 231
Robinson I, Robinson J R and Zund J D 1969 *J. Math. Mech.* **18** 881